L^p CHANGE OF VARIABLES INEQUALITIES ON MANIFOLDS

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ABSTRACT. We prove two-sided inequalities for the L^p-norm of a push-forward or pullback (with respect to an orientation-preserving diffeomorphism) on oriented volume and Riemannian manifolds. For a function or density on a volume manifold, these bounds depend only on the Jacobian determinant, which arises through the change of variables theorem. For an arbitrary differential form on a Riemannian manifold, however, these bounds are shown to depend on more general "spectral" properties of the diffeomorphism, using an appropriately defined notion of singular values. These spectral terms generalize the Jacobian determinant, which is recovered in the special cases of functions and densities (i.e., bottom and top forms).

1. INTRODUCTION

1.1. **Motivation.** One of the most important tools in integral calculus (both classically in $\mathbb{R}^n$ and on manifolds) is the change of variables theorem, which states that the integral of a density is invariant under pushforward and pullback by orientation-preserving diffeomorphisms. A closely related object is the Jacobian determinant, which describes how the volume form changes with respect to pushforwards and pullbacks.

This paper is motivated by the following, natural question:

> How do pushforwards and pullbacks affect the L^p-norm of a function or density, on an oriented volume manifold; or that of a differential form, on an oriented Riemannian manifold?

We prove two-sided inequalities for each of these cases, showing that the norms of the pushforward and pullback are controlled by "spectral" properties of the diffeomorphism, with respect to an appropriately defined notion of singular values. In the case of a function or density, the bounds simply depend on the Jacobian determinant of the diffeomorphism and that of its inverse; these can be thought of as, respectively, the product of the singular values and that of their reciprocals. For a general differential form, though, we encounter Jacobian determinant-like products that combine *both* singular

values and reciprocal singular values. In the L^p-norm, these products are also shown to involve the conjugate exponent q, which satisfies $1/p + 1/q = 1$.

1.2. **Main results.** The main results of this paper are summarized in the following theorems, which will be proved in the subsequent sections.

Theorem 1 (smooth functions). *Let M and N be oriented, n-dimensional manifolds with volume forms μ_M and μ_N, respectively, and let $\varphi\colon M \to N$ be an orientation-preserving diffeomorphism with Jacobian determinant $J(\mu_M, \mu_N)\varphi \in C^\infty(M)$. Then, for any function $u \in C^\infty(M)$ with compact support $\operatorname{supp} u$,*

$$\left\| \mathbf{1}_{\operatorname{supp} u} \left[J(\mu_M, \mu_N)\varphi \right]^{-1/p} \right\|_\infty^{-1} \|u\|_p$$
$$\leq \left\| u \circ \varphi^{-1} \right\|_p \leq \left\| \mathbf{1}_{\operatorname{supp} u} \left[J(\mu_M, \mu_N)\varphi \right]^{1/p} \right\|_\infty \|u\|_p,$$

for all $p \in [1, \infty]$.

If the Jacobian determinant is bounded uniformly on all of M (for example, if M and N are compact), then this immediately yields a uniform inequality,

$$\left\| \left[J(\mu_M, \mu_N)\varphi \right]^{-1/p} \right\|_\infty^{-1} \|u\|_p \leq \left\| u \circ \varphi^{-1} \right\|_p \leq \left\| \left[J(\mu_M, \mu_N)\varphi \right]^{1/p} \right\|_\infty \|u\|_p,$$

which holds for all compactly supported $u \in C^\infty(M)$. This implies that the map $u \mapsto u \circ \varphi^{-1}$ is bounded, and because smooth functions with compact support form a dense subset of $L^p(M)$, we can therefore extend this to the whole space. Hence, Theorem 1 has the following corollary.

Corollary 2 (L^p functions). *Let M and N be oriented, n-dimensional manifolds with volume forms μ_M and μ_N, respectively, and let $\varphi\colon M \to N$ be an orientation-preserving diffeomorphism with Jacobian determinant $J(\mu_M, \mu_N)\varphi \in C^\infty(M)$. If the Jacobian determinant is bounded uniformly on M, then for any $u \in L^p(M)$,*

$$\left\| \left[J(\mu_M, \mu_N)\varphi \right]^{-1/p} \right\|_\infty^{-1} \|u\|_p \leq \left\| u \circ \varphi^{-1} \right\|_p \leq \left\| \left[J(\mu_M, \mu_N)\varphi \right]^{1/p} \right\|_\infty \|u\|_p,$$

for all $p \in [1, \infty]$.

There is an analogous result for densities (i.e., n-forms) on a volume manifold. (Since densities are the Hodge dual of functions, and the L^p and L^q function spaces are dual to one another, it is perhaps not too surprising that the conjugate exponent q plays an important role here.)

Theorem 3 (smooth densities). *Let M and N be oriented, n-dimensional manifolds with volume forms μ_M and μ_N, respectively, and let $\varphi\colon M \to N$ be an orientation-preserving diffeomorphism with Jacobian determinant $J(\mu_M, \mu_N)\varphi \in C^\infty(M)$. Then, for any smooth density $u\mu_M \in \Omega^n(M)$, where $u \in C^\infty(M)$ has compact support $\operatorname{supp} u = \operatorname{supp} u\mu_M$,*

$$\left\| \mathbf{1}_{\operatorname{supp} u\mu_M} \left[J(\mu_M, \mu_N)\varphi \right]^{1/q} \right\|_\infty^{-1} \|u\mu_M\|_p$$
$$\leq \left\| \varphi_*(u\mu_M) \right\|_p \leq \left\| \mathbf{1}_{\operatorname{supp} u\mu_M} \left[J(\mu_M, \mu_N)\varphi \right]^{-1/q} \right\|_\infty \|u\mu_M\|_p,$$

for all $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$.

Corollary 4 (L^p densities). *Let M and N be oriented, n-dimensional manifolds with volume forms μ_M and μ_N, respectively, and let $\varphi \colon M \to N$ be an orientation-preserving diffeomorphism with Jacobian determinant $J(\mu_M, \mu_N)\varphi \in C^\infty(M)$. If the reciprocal $[J(\mu_M, \mu_N)\varphi]^{-1}$ is bounded uniformly on M, then for any density $u\mu_M \in L^p\Omega^n(M)$, where $u \in L^p(M)$,*

$$\left\| [J(\mu_M, \mu_N)\varphi]^{1/q} \right\|_\infty^{-1} \|u\mu_M\|_p$$
$$\leq \|\varphi_*(u\mu_M)\|_p \leq \left\| [J(\mu_M, \mu_N)\varphi]^{-1/q} \right\|_\infty \|u\mu_M\|_p,$$

for all $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$.

When M and N are Riemannian manifolds, however, we show that it is possible to obtain a much more general family of inequalities, which hold for arbitrary k-forms on M, where $k = 0, 1, \ldots, n$. In the special cases $k = 0$ and $k = n$, these inequalities are shown to recover the previous results for functions and densities, respectively. In order to state and prove these more general results, we will introduce a novel extension of the *singular values* of a mapping. Whereas singular values are traditionally defined only for linear maps between Euclidean vector spaces, we show that they can also be defined intrinsically for diffeomorphisms between Riemannian manifolds.

Theorem 5 (smooth k-forms). *Let (M, g_M) and (N, g_N) be oriented, n-dimensional Riemannian manifolds, and let $\varphi \colon M \to N$ be an orientation-preserving diffeomorphism with singular values $\alpha_1(x) \geq \cdots \geq \alpha_n(x) > 0$ at each $x \in M$. Then, for any smooth k-form $\omega \in \Omega^k(M)$, $k = 0, \ldots, n$, with compact support $\operatorname{supp}\omega$,*

$$\left\| \mathbf{1}_{\operatorname{supp}\omega} (\alpha_1 \cdots \alpha_k)^{1/q} (\alpha_{k+1} \cdots \alpha_n)^{-1/p} \right\|_\infty^{-1} \|\omega\|_p$$
$$\leq \|\varphi_*\omega\|_p \leq \left\| \mathbf{1}_{\operatorname{supp}\omega} (\alpha_1 \cdots \alpha_{n-k})^{1/p} (\alpha_{n-k+1} \cdots \alpha_n)^{-1/q} \right\|_\infty \|\omega\|_p,$$

for all $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$.

Corollary 6 (L^p k-forms). *Let (M, g_M) and (N, g_N) be oriented, n-dimensional Riemannian manifolds, and let $\varphi \colon M \to N$ be an orientation-preserving diffeomorphism with singular values $\alpha_1(x) \geq \cdots \geq \alpha_n(x) > 0$ at each $x \in M$. Given $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$, and some $k = 0, \ldots, n$, suppose that the product $(\alpha_1 \cdots \alpha_{n-k})^{1/p} (\alpha_{n-k+1} \cdots \alpha_n)^{-1/q}$ is bounded uniformly on M. Then, for any $\omega \in L^p\Omega^k(M)$,*

$$\left\| (\alpha_1 \cdots \alpha_k)^{1/q} (\alpha_{k+1} \cdots \alpha_n)^{-1/p} \right\|_\infty^{-1} \|\omega\|_p$$
$$\leq \|\varphi_*\omega\|_p \leq \left\| (\alpha_1 \cdots \alpha_{n-k})^{1/p} (\alpha_{n-k+1} \cdots \alpha_n)^{-1/q} \right\|_\infty \|\omega\|_p.$$

1.3. **Organization of the paper.** We begin, in the next section, by proving Theorem 1 and Theorem 3, which apply to functions and densities, respectively, on oriented volume manifolds.

In the subsequent section, we turn to the case of differential k-forms on a Riemannian manifold. Unlike the previous inequalities for volume manifolds, whose main ingredients are the change of variables theorem and Hölder's inequality, the Riemannian case requires completely new analytical tools, which we introduce and develop along the way. In particular, the proof of Theorem 5 depends crucially on the generalized definition of singular values for a diffeomorphism between Riemannian manifolds. (Note that this is distinct from the usual notion of *spectrum* for a Riemannian manifold, which typically refers to eigenvalues of the Laplacian [3].) This theorem, along with the new techniques required to state and prove it, are the most significant contributions of this paper.

The proof of Theorem 5 also depends on some facts from multilinear algebra, relating the singular values of a linear operator to the spectral norm of its induced map on alternating tensors. We provide supplementary technical details in Appendix A.

Finally, we mention that although these results only apply to pushforwards, as stated above, it is trivial to apply them to pullbacks as well. Using the definition $\varphi_* = \left(\varphi^{-1}\right)^*$, simply replace φ by φ^{-1} above, along with its corresponding Jacobian determinant and singular values. For completeness (and for the convenience of the reader), the pullback versions of these inequalities are stated in Appendix B.

2. Change of variables on volume manifolds

2.1. **Smooth functions with compact support.** Let M and N be oriented, n-dimensional manifolds, with volume forms μ_M and μ_N, respectively. If $\varphi \colon M \to N$ is an orientation-preserving diffeomorphism, recall that there exists a function $J\left(\mu_M, \mu_N\right)\varphi \in C^\infty(M)$, called the *Jacobian determinant*, such that $\varphi^*\mu_N = \left[J\left(\mu_M, \mu_N\right)\varphi\right]\mu_M$ (see, e.g., Abraham, Marsden, and Ratiu [1]). For any function $v \in C^\infty(N)$ with compact support, this implies the familiar change of variables formula

$$\int_N v\mu_N = \int_M \varphi^*\left(v\mu_N\right) = \int_M \left(v \circ \varphi\right)\left[J\left(\mu_M, \mu_N\right)\varphi\right]\mu_M.$$

Now, let $u \in C^\infty(M)$ be a function with compact support (denoted by $\operatorname{supp} u$). Then, for any $p \in [1, \infty)$, it follows that

$$\int_N \left|u \circ \varphi^{-1}\right|^p \mu_N = \int_M |u|^p \left[J\left(\mu_M, \mu_N\right)\varphi\right]\mu_M$$

$$\leq \left\|\mathbf{1}_{\operatorname{supp} u} J\left(\mu_M, \mu_N\right)\varphi\right\|_\infty \int_M |u|^p \mu_M,$$

by change of variables and Hölder's inequality. This immediately gives the L^p-norm upper bound

$$\left\|u \circ \varphi^{-1}\right\|_p \leq \left\|\mathbf{1}_{\operatorname{supp} u} \left[J\left(\mu_M, \mu_N\right)\varphi\right]^{1/p}\right\|_\infty \|u\|_p.$$

(Notice that this also holds for $p = \infty$.) To obtain the lower bound, we write

$$\int_M |u|^p \, \mu_M = \int_M \left(\left| u \circ \varphi^{-1} \right|^p \circ \varphi \right) \mu_M$$

$$= \int_M \varphi^* \left(\left| u \circ \varphi \right|^{-1} \varphi_* \mu_M \right)$$

$$= \int_M \varphi^* \left(\left| u \circ \varphi \right|^{-1} \left[J \left(\mu_N, \mu_M \right) \left(\varphi^{-1} \right) \right] \mu_N \right)$$

Since the Jacobian determinant satisfies the inverse identity

(1) $$J \left(\mu_N, \mu_M \right) \left(\varphi^{-1} \right) = \left[J \left(\mu_M, \mu_N \right) \varphi \right]^{-1} \circ \varphi^{-1},$$

it follows that

$$\int_M |u|^p \, \mu_M = \int_M \left[J \left(\mu_M, \mu_N \right) \varphi \right]^{-1} \varphi^* \left(\left| u \circ \varphi^{-1} \right|^p \mu_N \right)$$

$$\leq \left\| \mathbf{1}_{\operatorname{supp} u} \left[J \left(\mu_M, \mu_N \right) \varphi \right]^{-1} \right\|_\infty \int_N \left| u \circ \varphi^{-1} \right|^p \mu_N,$$

again using Hölder's inequality and change of variables. Thus,

$$\|u\|_p \leq \left\| \mathbf{1}_{\operatorname{supp} u} \left[J \left(\mu_M, \mu_N \right) \varphi \right]^{-1/p} \right\|_\infty \left\| u \circ \varphi^{-1} \right\|_p,$$

which rearranges to give the lower bound

$$\left\| u \circ \varphi^{-1} \right\|_p \geq \left\| \mathbf{1}_{\operatorname{supp} u} \left[J \left(\mu_M, \mu_N \right) \varphi \right]^{-1/p} \right\|_\infty^{-1} \|u\|_p.$$

In summary, we have now established the two-sided inequality,

$$\left\| \mathbf{1}_{\operatorname{supp} u} \left[J \left(\mu_M, \mu_N \right) \varphi \right]^{-1/p} \right\|_\infty^{-1} \|u\|_p$$

$$\leq \left\| u \circ \varphi^{-1} \right\|_p \leq \left\| \mathbf{1}_{\operatorname{supp} u} \left[J \left(\mu_M, \mu_N \right) \varphi \right]^{1/p} \right\|_\infty \|u\|_p,$$

which completes the proof of Theorem 1. $\qquad\square$

Remark. If φ is volume-preserving (at least on the support of u), then it also preserves the L^p-norm for all p. Indeed, if $J \left(\mu_M, \mu_N \right) \varphi = 1$, then the inequality simply becomes $\|u\|_p \leq \left\| u \circ \varphi^{-1} \right\|_p \leq \|u\|_p$, and thus $\left\| u \circ \varphi^{-1} \right\|_p = \|u\|_p$, as expected. This equality is also seen to hold for arbitrary diffeomorphisms φ (i.e., not necessarily volume-preserving) when $p = \infty$.

2.2. Smooth densities with compact support.
Any smooth density on N with compact support can be written as $v\mu_N$, where $v \in C^\infty(N)$ is a compactly-supported function. Now, since the Hodge star operator $\star_N$ is an isometry, with $\star_N \mu_N = 1$, the pointwise norm of this density satisfies $|v\mu_N| = |\star_N (v\mu_N)| = |v|$ (and likewise for $\star_M$). In particular, this implies the identity

$$|v\mu_N|^p \circ \varphi = |v|^p \circ \varphi = |v \circ \varphi|^p = |(v \circ \varphi) \mu_M|^p,$$

Therefore,

(2) $$|\varphi^* (v\mu_N)|^p = |(v \circ \varphi) \left[J \left(\mu_M, \mu_N \right) \varphi \right] \mu_M|^p$$

$$= (|v\mu_N|^p \circ \varphi) \left[J \left(\mu_M, \mu_N \right) \varphi \right]^p.$$

Now, consider the pushforward of the density $u\mu_M$, where $u \in C^\infty(M)$ has compact support $\operatorname{supp} u = \operatorname{supp} u\mu_M$. Using change of variables,

$$
\begin{aligned}
\int_N |\varphi_*(u\mu_M)|^p \,\mu_N &= \int_M \left(|\varphi_*(u\mu_M)|^p \circ \varphi\right) \left[J(\mu_M, \mu_N)\,\varphi\right] \mu_M \\
&= \int_M |u\mu_M|^p \left[J(\mu_M, \mu_N)\,\varphi\right]^{1-p} \mu_M \\
&\leq \left\|\mathbf{1}_{\operatorname{supp} u\mu_M}\left[J(\mu_M, \mu_N)\,\varphi\right]^{1-p}\right\|_\infty \int_M |u\mu_M|^p \,\mu_M,
\end{aligned}
$$

where the last two lines use the identity (2) and Hölder's inequality, respectively. Taking the conjugate exponent q such that $1/p + 1/q = 1$, observe that $(1-p)/p = -1/q$. Therefore, we can write the upper bound

$$
\|\varphi_*(u\mu_M)\|_p \leq \left\|\mathbf{1}_{\operatorname{supp} u\mu_M}\left[J(\mu_M, \mu_N)\,\varphi\right]^{-1/q}\right\|_\infty \|u\mu_M\|_p.
$$

For the lower bound, we begin by using the identity (2) to write

$$
|u\mu_M|^p = |\varphi^*\varphi_*(u\mu_M)|^p = \left(|\varphi_*(u\mu_M)|^p \circ \varphi\right) \left[J(\mu_M, \mu_N)\,\varphi\right]^p,
$$

and thus

$$
\begin{aligned}
\int_M |u\mu_M|^p \,\mu_M &= \int_M \left(|\varphi_*(u\mu_M)|^p \circ \varphi\right) \left[J(\mu_M, \mu_N)\,\varphi\right]^p \mu_M \\
&= \int_M \left[J(\mu_M, \mu_N)\,\varphi\right]^p \varphi^*\left(|\varphi_*(u\mu_M)|^p \varphi_*\mu_M\right) \\
&= \int_M \left[J(\mu_M, \mu_N)\,\varphi\right]^{p-1} \varphi^*\left(|\varphi_*(u\mu_M)|^p \mu_N\right) \\
&\leq \left\|\mathbf{1}_{\operatorname{supp} u\mu_M}\left[J(\mu_M, \mu_N)\,\varphi\right]^{p-1}\right\|_\infty \int_N |\varphi_*(u\mu_M)|^p \,\mu_N,
\end{aligned}
$$

by the Jacobian inverse identity (1), Hölder's inequality, and change of variables. Therefore,

$$
\|u\mu_M\|_p \leq \left\|\mathbf{1}_{\operatorname{supp} u\mu_M}\left[J(\mu_M, \mu_N)\,\varphi\right]^{1/q}\right\|_\infty \|\varphi_*(u\mu_M)\|_p,
$$

so rearranging, we have the lower bound

$$
\|\varphi_*(u\mu_M)\|_p \geq \left\|\mathbf{1}_{\operatorname{supp} u\mu_M}\left[J(\mu_M, \mu_N)\,\varphi\right]^{1/q}\right\|_\infty^{-1} \|u\mu_M\|_p.
$$

Hence, we have shown the two-sided inequality

$$
\begin{aligned}
\left\|\mathbf{1}_{\operatorname{supp} u\mu_M}\right.&\left.\left[J(\mu_M, \mu_N)\,\varphi\right]^{1/q}\right\|_\infty^{-1} \|u\mu_M\|_p \\
&\leq \|\varphi_*(u\mu_M)\|_p \leq \left\|\mathbf{1}_{\operatorname{supp} u\mu_M}\left[J(\mu_M, \mu_N)\,\varphi\right]^{-1/q}\right\|_\infty \|u\mu_M\|_p,
\end{aligned}
$$

which completes the proof of Theorem 3. $\square$

Remark. For densities, the L^p-norm is preserved if either φ is volume-preserving (at least on $\operatorname{supp} u\mu_M$) or when $p = 1$, $q = \infty$. In these cases, the inequality becomes $\|u\mu_M\|_p \leq \|\varphi_*(u\mu_M)\|_p \leq \|u\mu_M\|_p$, and thus $\|\varphi_*(u\mu_M)\|_p = \|u\mu_M\|_p$.

3. Change of variables on Riemannian manifolds

3.1. **Singular values of a diffeomorphism.** In order to generalize this result to differential forms, we suppose now that (M, g_M) and (N, g_N) are oriented, n-dimensional Riemannian manifolds, with μ_M and μ_N denoting their respective Riemannian volume forms. As before, let $\varphi \colon M \to N$ be an orientation-preserving diffeomorphism. Given a point $x \in M$, let $\{e_1, \ldots, e_n\}$ be a positively-oriented, g_M-orthonormal basis of the tangent space $T_x M$, and let $\{f_1, \ldots, f_n\}$ be a positively-oriented, g_N-orthonormal basis of $T_{\varphi(x)} N$. With respect to these bases, the tangent map $T_x \varphi \colon T_x M \to T_{\varphi(x)} N$ can be represented by an $n \times n$ matrix Φ. Since φ is a diffeomorphism, the matrix Φ has n positive singular values, which we write

$$\alpha_1(x) \geq \cdots \geq \alpha_n(x) > 0.$$

The singular values of Φ are orthogonally invariant, so they are independent of the choice of orthonormal basis, and thus are an intrinsic property of the diffeomorphism. Therefore, we refer to these as the *singular values of φ at x*. It follows that the pullback of the volume form on N is

$$\varphi^* \mu_N = (\det \Phi) \, \mu_M = (\alpha_1 \cdots \alpha_n) \, \mu_M,$$

so the Jacobian determinant is simply the product of the singular values $J(\mu_M, \mu_N) \varphi = \alpha_1 \cdots \alpha_n$.

Similarly, the inverse map $T_{\varphi(x)} (\varphi^{-1}) : T_{\varphi(x)} N \to T_x M$ is represented by the inverse matrix Φ^{-1}, whose singular values are the reciprocals of those for Φ. Hence, we write the singular values of φ^{-1} at $\varphi(x)$ as

$$\beta_1(\varphi(x)) \geq \cdots \geq \beta_n(\varphi(x)) > 0,$$

which satisfy $\beta_i(\varphi(x)) = \alpha_{n-i+1}(x)^{-1}$, i.e., $\beta_i = \alpha_{n-i+1}^{-1} \circ \varphi^{-1}$, for $i = 1, \ldots, n$. Consequently, the pushforward of the volume form on M is

$$\varphi_* \mu_M = (\det \Phi^{-1}) \, \mu_N = (\beta_1 \cdots \beta_n) \, \mu_N.$$

Therefore, the Jacobian determinant is

$$J(\mu_N, \mu_M)(\varphi^{-1})$$
$$= \beta_1 \cdots \beta_n = (\alpha_1 \cdots \alpha_n)^{-1} \circ \varphi^{-1} = [J(\mu_M, \mu_N) \varphi]^{-1} \circ \varphi^{-1},$$

so we recover the usual identity (1).

Remark. Using the well-known "minimax" and "maximin" characterizations of singular values, it is also possible to write

$$\alpha_i(x) = \min_{\substack{S \subset \mathbb{R}^n \\ \dim S = n-i+1}} \max_{0 \neq X \in S} \frac{|\Phi X|}{|X|} = \max_{\substack{S \subset \mathbb{R}^n \\ \dim S = i}} \min_{0 \neq X \in S} \frac{|\Phi X|}{|X|}$$
$$= \min_{\substack{S \subset T_x M \\ \dim S = n-i+1}} \max_{0 \neq X \in S} \frac{|T_x \varphi(X)|}{|X|} = \max_{\substack{S \subset T_x M \\ \dim S = i}} \min_{0 \neq X \in S} \frac{|T_x \varphi(X)|}{|X|}.$$

This can be taken as an alternative, basis-independent definition for the singular values of a diffeomorphism, consistent with the previous one. This also provides another way to see that $\beta_i = \alpha_{n-i+1} \circ \varphi^{-1}$, since

$$
\begin{aligned}
\beta_i\left(\varphi(x)\right) &= \min_{\substack{S \subset T_{\varphi(x)}N \\ \dim S = n-i+1}} \max_{0 \neq Y \in S} \frac{\left|T_{\varphi(x)}\left(\varphi^{-1}\right)(Y)\right|}{|Y|} \\
&= \min_{\substack{S \subset T_x M \\ \dim S = n-i+1}} \max_{0 \neq X \in S} \frac{|X|}{|T_x \varphi(X)|} \\
&= \left(\max_{\substack{S \subset T_x M \\ \dim S = n-i+1}} \min_{0 \neq X \in S} \frac{|T_x \varphi(X)|}{|X|}\right)^{-1} \\
&= \alpha_{n-i+1}(x)^{-1}.
\end{aligned}
$$

3.2. **Pointwise inequalities for the spectral norm.** Now, given a smooth k-form $\omega \in \Omega^k(M)$ with compact support, recall that the spectral norm of ω at a point $x \in M$ is defined by

$$
|\omega| = \max_{0 \neq X_1, \ldots, X_k \in T_x M} \frac{|\omega\left(X_1, \ldots, X_k\right)|}{|X_1| \cdots |X_k|},
$$

where $|X_i| = g_M\left(X_i, X_i\right)^{1/2}$ denotes the length of the tangent vector X_i. With this definition, the L^p-norm on $\Omega^k(M)$ is simply $\|\omega\|_p = \int_M |\omega|^p \, \mu_M$, as usual. Likewise, for $\eta \in \Omega^k(N)$, at each point $y \in N$ we have

$$
|\eta| = \max_{0 \neq Y_1, \ldots, Y_k \in T_y N} \frac{|\eta\left(Y_1, \ldots, Y_k\right)|}{|Y_1| \cdots |Y_k|},
$$

where here $|Y_i| = g_N\left(Y_i, Y_i\right)^{1/2}$, and $\|\eta\|_p = \int_N |\eta|^p \, \mu_N$.

To prove Theorem 5, we begin by stating pointwise bounds for the pullback and pushforward, in terms of the singular values of φ and φ^{-1}. Since ω and η are k-linear and totally antisymmetric at each point, it is straightforward to show that

$$
\begin{aligned}
(3) \qquad & |\varphi^* \eta| \leq \alpha_1 \cdots \alpha_k \left(|\eta| \circ \varphi\right) \quad = \left[\left(\beta_{n-k+1} \cdots \beta_n\right)^{-1} |\eta|\right] \circ \varphi \\
(4) \qquad & |\varphi_* \omega| \leq \beta_1 \cdots \beta_k \left(|\omega| \circ \varphi^{-1}\right) = \left[\left(\alpha_{n-k+1} \cdots \alpha_n\right)^{-1} |\omega|\right] \circ \varphi^{-1}.
\end{aligned}
$$

That is, the pullback of a k-form is controlled by the product of the k largest singular values of φ, while the pushforward is controlled by the product of the k largest singular values of φ^{-1}. Further discussion of these pointwise inequalities, as well as some background and details on their derivation, is given in Appendix A.

3.3. **Change of variables for k-forms.** Using change of variables, the pointwise inequality (4) for the pushforward, and Hölder's inequality,

$$\int_N |\varphi_*\omega|^p \, \mu_N = \int_M \varphi^* \left(|\varphi_*\omega|^p \, \mu_N\right)$$

$$= \int_M \left(|\varphi_*\omega|^p \circ \varphi\right) \varphi^* \mu_N$$

$$\leq \int_M \left[(\alpha_{n-k+1} \cdots \alpha_n)^{-1} |\omega|\right]^p (\alpha_1 \cdots \alpha_n) \, \mu_M$$

$$= \int_M (\alpha_1 \cdots \alpha_{n-k}) (\alpha_{n-k+1} \cdots \alpha_n)^{1-p} |\omega|^p \, \mu_M$$

$$\leq \left\|\mathbf{1}_{\operatorname{supp}\omega} (\alpha_1 \cdots \alpha_{n-k}) (\alpha_{n-k+1} \cdots \alpha_n)^{1-p}\right\|_\infty \int_M |\omega|^p \, \mu_M.$$

Hence, we immediately obtain the upper bound

$$\|\varphi_*\omega\|_p \leq \left\|\mathbf{1}_{\operatorname{supp}\omega} (\alpha_1 \cdots \alpha_{n-k})^{1/p} (\alpha_{n-k+1} \cdots \alpha_n)^{-1/q}\right\|_\infty \|\omega\|_p.$$

To get the lower bound, we begin by using the pointwise inequality (3) for the pullback to write

$$|\omega| = |\varphi^*\varphi_*\omega| \leq \alpha_1 \cdots \alpha_k \left(|\varphi_*\omega| \circ \varphi\right).$$

Thus, using change of variables and Hölder's inequality once again,

$$\int_M |\omega|^p \, \mu_M \leq \int_M (\alpha_1 \cdots \alpha_k)^p \left(|\varphi_*\omega|^p \circ \varphi\right) \mu_M$$

$$= \int_M \varphi^* \left(\left[(\alpha_1 \cdots \alpha_k)^p \circ \varphi^{-1}\right] |\varphi_*\omega|^p \, \varphi_*\mu_M\right)$$

$$= \int_M \varphi^* \left(\left[(\alpha_1 \cdots \alpha_k)^p \circ \varphi^{-1}\right] |\varphi_*\omega|^p \left[(\alpha_1 \cdots \alpha_n)^{-1} \circ \varphi^{-1}\right] \mu_N\right)$$

$$= \int_M (\alpha_1 \cdots \alpha_k)^{p-1} (\alpha_{k+1} \cdots \alpha_n)^{-1} \varphi^* \left(|\varphi_*\omega|^p \, \mu_N\right)$$

$$\leq \left\|\mathbf{1}_{\operatorname{supp}\omega} (\alpha_1 \cdots \alpha_k)^{p-1} (\alpha_{k+1} \cdots \alpha_n)^{-1}\right\|_\infty \int_N |\varphi_*\omega|^p \, \mu_N,$$

so

$$\|\omega\|_p \leq \left\|\mathbf{1}_{\operatorname{supp}\omega} (\alpha_1 \cdots \alpha_k)^{1/q} (\alpha_{k+1} \cdots \alpha_n)^{-1/p}\right\|_\infty \|\varphi_*\omega\|_p.$$

Hence, this implies the lower bound

$$\|\varphi_*\omega\|_p \geq \left\|\mathbf{1}_{\operatorname{supp}\omega} (\alpha_1 \cdots \alpha_k)^{1/q} (\alpha_{k+1} \cdots \alpha_n)^{-1/p}\right\|_\infty^{-1} \|\omega\|_p.$$

Combining the upper and lower bounds, we have finally established the two-sided inequality

$$\left\|\mathbf{1}_{\operatorname{supp}\omega} (\alpha_1 \cdots \alpha_k)^{1/q} (\alpha_{k+1} \cdots \alpha_n)^{-1/p}\right\|_\infty^{-1} \|\omega\|_p$$

$$\leq \|\varphi_*\omega\|_p \leq \left\|\mathbf{1}_{\operatorname{supp}\omega} (\alpha_1 \cdots \alpha_{n-k})^{1/p} (\alpha_{n-k+1} \cdots \alpha_n)^{-1/q}\right\|_\infty \|\omega\|_p,$$

which completes the proof of Theorem 5. $\qquad\square$

Remark. If φ is an isometry (at least on $\operatorname{supp}\omega$), then it preserves the L^p-norm for all p. Indeed, isometry implies that the matrix Φ is orthogonal at every $x \in M$, so the singular values are $\alpha_1 = \cdots = \alpha_n = 1$. Therefore, the inequality becomes $\|\omega\|_p \leq \|\varphi_*\omega\|_p \leq \|\omega\|_p$, and hence $\|\varphi_*\omega\|_p = \|\omega\|_p$.

Note that for $k = 0$, this simply reduces to Theorem 1, so the L^p-norm is preserved whenever φ is volume-preserving (since $\alpha_1 \cdots \alpha_n = 1$) or when $p = \infty$, $q = 1$. Similarly, for $k = n$, this reduces to Theorem 3, so the L^p-norm is preserved whenever φ is volume-preserving or when $p = 1$, $q = \infty$. More generally, though, for $0 < k < n$, volume preservation is *not* sufficient: it merely implies that the product of all n singular values equals 1, but it does not imply that this also holds for products of the k largest or smallest singular values.

Appendix A. Alternating tensors and pointwise inequalities

The pointwise inequalities (3) and (4) are a consequence of some facts from multilinear algebra regarding alternating tensors. Consider the case where Φ is a linear isomorphism on $\mathbb{R}^n$ with singular values $\alpha_1 \geq \cdots \geq \alpha_n > 0$. Associated to Φ, there is a so-called *compound* (or *induced operator*) on the kth exterior power, defined by

$$C_k(\Phi) \colon \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n, \quad X_1 \wedge \cdots \wedge X_k \mapsto \Phi X_1 \wedge \cdots \wedge \Phi X_k.$$

Due to the total antisymmetry of the exterior product, it can be shown that $|C_k(\Phi)| = \alpha_1 \cdots \alpha_k$ (see, for example, Andresen and Marcus [2], Marcus and Andresen [5], Li [4]). This result is derived by considering the representation of $C_k(\Phi)$ as an $\binom{n}{k} \times \binom{n}{k}$ matrix, whose entries are the corresponding $k \times k$ minors of Φ, and showing that its singular values are precisely the k-fold products of singular values of Φ. Hence the largest singular value of $C_k(\Phi)$ is $\alpha_1 \cdots \alpha_k$, the product of the k largest singular values of Φ. (This basic fact about singular values of the compound matrix was used at least as early as Weyl [6], where it was instrumental in proving the famous Weyl inequalities relating eigenvalues to singular values.) It follows that, if A is any alternating k-linear form on $\mathbb{R}^n$, we have

$$\bigl(A \circ C_k(\Phi)\bigr)(X_1, \ldots, X_k) = A\bigl(\Phi X_1, \ldots, \Phi X_k\bigr),$$

which satisfies the spectral norm inequality

$$|A \circ C_k(\Phi)| \leq |A|\,|C_k(\Phi)| = \alpha_1 \cdots \alpha_k\,|A|.$$

Now, as before, suppose that Φ is the matrix representation of $T_x\varphi \colon T_xM \to T_{\varphi(x)}N$ at some point $x \in M$, relative to a positively-oriented, g_M-orthonormal basis at x and a positively-oriented, g_N-orthonormal basis at $\varphi(x)$. Then the compound matrix $C_k(\Phi)$ induces a linear map $\bigwedge^k T_xM \to \bigwedge^k T_{\varphi(x)}N$, and has $|C_k(\Phi)| = \alpha_1(x) \cdots \alpha_k(x)$. If A is the corresponding tensor representation of $\eta \in \Omega^k(N)$ at $\varphi(x) \in N$, then $\varphi^*\eta$ is given by $A \circ C_k(\Phi)$. Thus,

$$|\varphi^*\eta| = |A \circ C_k(\Phi)| \leq \alpha_1(x) \cdots \alpha_k(x)\,|A| = \alpha_1(x) \cdots \alpha_k(x)\,|\eta|\,(\varphi(x)),$$

as stated in (3). Likewise, if B represents $\omega \in \Omega^k(M)$ at $x \in M$, then

$$|\varphi_* \omega| = \left|B \circ C_k\left(\Phi^{-1}\right)\right|$$
$$\leq \beta_1\left(\varphi(x)\right) \cdots \beta_k\left(\varphi(x)\right)|B| = \beta_1\left(\varphi(x)\right) \cdots \beta_k\left(\varphi(x)\right)|\omega|(x),$$

as in (4). (As with the singular values themselves, these statements are independent of the particular choice of basis at $x \in M$ or $\varphi(x) \in N$.)

Notice that, for $k = 0$, there is no contribution from the singular values, and we just obtain the equalities $|v \circ \varphi| = |v| \circ \varphi$ and $\left|u \circ \varphi^{-1}\right| = |u| \circ \varphi^{-1}$. This is why, in the scalar case, it was possible to simply apply the Jacobian determinant formula to get

$$\varphi^*\left(|v|^p \mu_N\right) = \left(|v|^p \circ \varphi\right)\varphi^* \mu_N = |v \circ \varphi|^p\left[J\left(\mu_M, \mu_N\right)\varphi\right]\mu_M,$$

while slightly more care was necessary for the density and k-form cases.

APPENDIX B. PULLBACK INEQUALITIES

This appendix contains variants of the main results, stated for pullbacks rather than pushforwards. As mentioned in the introduction, these corollaries follow trivially from the pushforward results given elsewhere in the paper, simply by replacing φ by φ^{-1}.

Corollary 7 (smooth functions). *Let M and N be oriented, n-dimensional manifolds with volume forms μ_M and μ_N, respectively, and let $\varphi \colon M \to N$ be an orientation-preserving diffeomorphism, whose inverse φ^{-1} has the Jacobian determinant $J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right) \in C^\infty(N)$. Then, for any function $v \in C^\infty(N)$ with compact support $\operatorname{supp} v$,*

$$\left\|\mathbf{1}_{\operatorname{supp} v}\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{-1/p}\right\|_\infty^{-1}\|v\|_p$$
$$\leq \|v \circ \varphi\|_p \leq \left\|\mathbf{1}_{\operatorname{supp} v}\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{1/p}\right\|_\infty \|v\|_p,$$

for all $p \in [1, \infty]$.

Corollary 8 (L^p functions). *Let M and N be oriented, n-dimensional manifolds with volume forms μ_M and μ_N, respectively, and let $\varphi \colon M \to N$ be an orientation-preserving diffeomorphism, whose inverse φ^{-1} has the Jacobian determinant $J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right) \in C^\infty(N)$. If this inverse Jacobian determinant is bounded uniformly on N, then for any $v \in L^p(N)$,*

$$\left\|\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{-1/p}\right\|_\infty^{-1}\|v\|_p$$
$$\leq \|v \circ \varphi\|_p \leq \left\|\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{1/p}\right\|_\infty \|v\|_p,$$

for all $p \in [1, \infty]$.

Remark. By the Jacobian determinant inverse identity (1), it follows that $J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)$ is bounded uniformly on N if and only if the reciprocal

$[J\left(\mu_M, \mu_N\right)\varphi]^{-1}$ is bounded uniformly on M. Hence, this inequality can also be written as

$$\left\|[J\left(\mu_M, \mu_N\right)\varphi]^{1/p}\right\|_\infty^{-1}\|v\|_p \leq \|v \circ \varphi\|_p \leq \left\|[J\left(\mu_M, \mu_N\right)\varphi]^{-1/p}\right\|_\infty \|v\|_p.$$

Corollary 9 (smooth densities). *Let M and N be oriented, n-dimensional manifolds with volume forms μ_M and μ_N, respectively, and let $\varphi\colon M \to N$ be an orientation-preserving diffeomorphism, whose inverse φ^{-1} has the Jacobian determinant $J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right) \in C^\infty(N)$. Then, for any smooth density $v\mu_N \in \Omega^n(N)$, where $v \in C^\infty(N)$ has compact support $\operatorname{supp} v = \operatorname{supp} v\mu_N$,*

$$\left\|\mathbf{1}_{\operatorname{supp} v\mu_N}\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{1/q}\right\|_\infty^{-1}\|v\mu_N\|_p$$
$$\leq \|\varphi^*(v\mu_N)\|_p \leq \left\|\mathbf{1}_{\operatorname{supp} v\mu_N}\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{-1/q}\right\|_\infty \|v\mu_N\|_p,$$

for all $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$.

Corollary 10 (L^p densities). *Let M and N be oriented, n-dimensional manifolds with volume forms μ_M and μ_N, respectively, and let $\varphi\colon M \to N$ be an orientation-preserving diffeomorphism, whose inverse φ^{-1} has the Jacobian determinant $J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right) \in C^\infty(N)$. If the reciprocal $\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{-1}$ is bounded uniformly on N, then for any density $v\mu_N \in L^p\Omega^n(N)$, where $v \in L^p(N)$,*

$$\left\|\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{1/q}\right\|_\infty^{-1}\|v\mu_N\|_p$$
$$\leq \|\varphi^*(v\mu_N)\|_p \leq \left\|\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{-1/q}\right\|_\infty \|v\mu_N\|_p,$$

for all $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$.

Remark. Again, by the Jacobian determinant inverse identity (1), the reciprocal $\left[J\left(\mu_N, \mu_M\right)\left(\varphi^{-1}\right)\right]^{-1}$ is bounded uniformly on N if and only if $J\left(\mu_M, \mu_N\right)\varphi$ is bounded uniformly on M. Hence, this inequality can also be written as

$$\left\|[J\left(\mu_M, \mu_N\right)\varphi]^{-1/q}\right\|_\infty^{-1}\|v\mu_N\|_p$$
$$\leq \|\varphi^*(v\mu_N)\|_p \leq \left\|[J\left(\mu_M, \mu_N\right)\varphi]^{1/q}\right\|_\infty \|v\mu_N\|_p.$$

Corollary 11 (smooth k-forms). *Let (M, g_M) and (N, g_N) be oriented, n-dimensional Riemannian manifolds, and let $\varphi\colon M \to N$ be an orientation-preserving diffeomorphism, whose inverse has singular values $\beta_1(y) \geq \cdots \geq \beta_n(y) > 0$ at each $y \in N$. Then, for any smooth k-form $\eta \in \Omega^k(N)$, $k = 0, \ldots, n$, with compact support $\operatorname{supp} \eta$,*

$$\left\|\mathbf{1}_{\operatorname{supp} \eta}\left(\beta_1 \cdots \beta_k\right)^{1/q}\left(\beta_{k+1} \cdots \beta_n\right)^{-1/p}\right\|_\infty^{-1}\|\eta\|_p$$
$$\leq \|\varphi^*\eta\|_p \leq \left\|\mathbf{1}_{\operatorname{supp} \eta}\left(\beta_1 \cdots \beta_{n-k}\right)^{1/p}\left(\beta_{n-k+1} \cdots \beta_n\right)^{-1/q}\right\|_\infty \|\eta\|_p,$$

for all $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$.

Corollary 12 (L^p k-forms)**.** *Let (M, g_M) and (N, g_N) be oriented, n-dimensional Riemannian manifolds, and let $\varphi \colon M \to N$ be an orientation-preserving diffeomorphism, whose inverse has singular values $\beta_1(y) \geq \cdots \geq \beta_n(y) > 0$ at each $y \in N$. Given $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$, and some $k = 0, \ldots, n$, suppose that the product $(\beta_1 \cdots \beta_{n-k})^{1/p} (\beta_{n-k+1} \cdots \beta_n)^{-1/q}$ is bounded uniformly on N. Then, for any $\eta \in L^p \Omega^k(N)$,*

$$\left\| (\beta_1 \cdots \beta_k)^{1/q} (\beta_{k+1} \cdots \beta_n)^{-1/p} \right\|_\infty^{-1} \|\eta\|_p$$
$$\leq \|\varphi^* \eta\|_p \leq \left\| (\beta_1 \cdots \beta_{n-k})^{1/p} (\beta_{n-k+1} \cdots \beta_n)^{-1/q} \right\|_\infty \|\eta\|_p.$$

Remark. Using the singular value inverse identity $\beta_i = \alpha_{n-i+1}^{-1} \circ \varphi^{-1}$, it follows that $(\beta_1 \cdots \beta_{n-k})^{1/p} (\beta_{n-k+1} \cdots \beta_n)^{-1/q}$ is bounded uniformly on N if and only if $(\alpha_1 \cdots \alpha_k)^{-1/q} (\alpha_{k+1} \cdots \alpha_n)^{1/p}$ is bounded uniformly on M. Therefore, this inequality can also be written as

$$\left\| (\alpha_1 \cdots \alpha_{n-k})^{-1/p} (\alpha_{n-k+1} \cdots \alpha_n)^{1/q} \right\|_\infty^{-1} \|\eta\|_p$$
$$\leq \|\varphi^* \eta\|_p \leq \left\| (\alpha_1 \cdots \alpha_k)^{-1/q} (\alpha_{k+1} \cdots \alpha_n)^{1/p} \right\|_\infty \|\eta\|_p.$$

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